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Invariant Manifolds for Impulsive Equations and Nonuniform Polynomial Dichotomies

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Abstract For impulsive differential equations in Banach spaces, we construct stable and unstable invariant manifolds for sufficiently small perturbations of a polynomial dichotomy. We also consider the general case of *nonuniform* polynomial dichotomies. Moreover, we introduce the notions of polynomial Lyapunov exponent and of regularity coefficient for a linear impulsive differential equation, and we show that when the Lyapunov exponent never vanishes the linear equation admits a nonuniform polynomial dichotomy.

Keywords Stable invariant manifolds · Nonuniform polynomial dichotomies · Impulsive differential equations

1 Introduction

1.1 Invariant Manifolds for Impulsive Equations

Our main objective is to construct stable and unstable invariant manifolds for impulsive differential equations in a Banach space, obtained from sufficiently small perturbations of a

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polynomial dichotomy. In addition, we consider the general case of *nonuniform* polynomial dichotomies.

An impulsive differential equation corresponds to a smooth evolution of a dynamics that at certain times changes instantaneously, corresponding to impulses in the smooth system. There are many applications of these equations to mechanical and natural phenomena involving abrupt changes, including in physics, chemistry, biology, control theory, and robotics. We refer to the books [1, 14, 20] for detailed descriptions and for an extensive list of references.

The notion of exponential dichotomy, essentially introduced in seminal work of Perron [16], together with some of its variants and generalizations, plays a central role in a substantial part of the theory of differential equations and dynamical systems. In particular, it causes the existence of stable and unstable invariant manifolds under sufficiently small perturbations of the dichotomy. It turns out that the classical notion of (uniform) exponential dichotomy is very stringent for the dynamics and it is of interest to look for more general types of hyperbolic behavior. This is precisely what happens with the notion of nonuniform exponential behavior. We refer to [5] for a detailed exposition of the theory, which goes back to the landmark works of Oseledets [15] and particularly Pesin [17]. It is an important part of the general theory of dynamical systems and a principal tool in the study of stochastic behavior. In particular, the notion of nonuniform hyperbolicity plays an important role in the construction of stable and unstable invariant manifolds (see [17–19]). We refer to [5, 6] for related detailed discussions.

In the theory of impulsive equations, the notion of exponential dichotomy also plays an important role (see for example [2] and the references therein). For the construction of stable and unstable invariant manifolds in the case of impulsive equations we refer to [20] for perturbations of a uniform exponential dichotomy, and to [8] for a corresponding construction for perturbations of a nonuniform exponential dichotomy.

1.2 Nonuniform Polynomial Dichotomies

Recently, the notion of nonuniform polynomial dichotomy was introduced independently in [7] and [9], in somewhat distinct forms, respectively in the cases of continuous and discrete time. In this case the rates of expansion and contraction vary polynomially. In particular, it was shown in [7] that the notion of nonuniform polynomial dichotomy occurs naturally, in the sense that it can be deduced from the nonvanishing of a certain polynomial Lyapunov exponent (see Theorem 4 for a version of this result in the case of impulsive equations).

It should be noted that the notions of exponential dichotomy and polynomial dichotomy are particular cases of a general notion of dichotomy with asymptotic rates of the form $e^{c\rho(t)}$, for some arbitrary increasing function $\rho\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\rho(0) = 0$ (taking $c < 0$ in the stable direction and $c > 0$ in the unstable direction). The usual exponential behavior corresponds to take $\rho(t) = t$, and the polynomial behavior may be obtained by taking $\rho(t) = \log(1 + t)$. We point out that it is easy to construct examples of (linear) differential equations $x' = A(t)x$ for which all Lyapunov exponents are infinite, and in this situation one is not able to apply the existing stability theory. However, this may be due to the existence of other asymptotic behaviors, such as polynomial behavior, and it turns that these are also quite frequent (we refer to [7] for a detailed discussion). On this respect, we would like to point out that even in the trivial case of linear differential equations with constant coefficients only the real parts of the eigenvalues matter for the study of stability (since the nontrivial Jordan blocks leading to polynomial terms do not affect the stability determined by a set of eigenvalues with nonzero real parts). Analogously, in our context it makes sense to separate large classes of asymptotic behaviors (such as exponential and polynomial, the last one corresponding in the trivial case of constant coefficients to the existence of eigenvalues with zero real part and possibly nontrivial Jordan blocks). Moreover, it turns out that some techniques already available in the case of exponential behavior cannot be applied directly or without new ideas (again we refer to [7] for a related discussion).

Here we consider the notion of nonuniform polynomial dichotomy in the more general setting of impulsive differential equations. In particular, we construct stable and unstable invariant manifolds for sufficiently small perturbations of a nonuniform polynomial dichotomy. We also introduce the notions of polynomial Lyapunov exponent and of regularity coefficient for a linear impulsive differential equation, now in a finite-dimensional space, and we show that when the Lyapunov exponent never vanishes the linear equation admits a nonuniform polynomial dichotomy.

Incidentally, in [9] the authors claim the construction of invariant stable manifolds for difference equations obtained from sufficiently small perturbations of nonuniform polynomial dichotomies. The dichotomies are defined with respect to the original norm, of course as one should. However, they then use the norms $\|(x, y)\|_n = \|x\| + \|y\|$ (the notation $\|\cdot\|_n$ is our own), with $(x, y) \in E_n \times F_n$ where E_n and F_n are the stable and unstable subspaces at time n, but apparently failed to notice that the estimates in the notion of dichotomy may change (indeed they continue to use the polynomial estimates with respect to the original norm).

1.3 Further Developments and Statistical Physics

We also would like to discuss several potential developments of our results, in particular in connection to statistical physics and the study of stochastic properties. It is well known that systems with nonzero Lyapunov exponents exhibit a wide range of strong stochastic properties (see for example [3, 5] and the references therein). For the discussion of some related open problems also concerning the relation to statistical physics, we recommend [10]. In addition, there is a large class of systems with singularities also exhibiting a wide range of stochastic properties. In particular, we can consider those preserving volume of which billiards are the main example (we refer to [4, 12, 13] for details). This involves constructing stable and unstable manifolds at every so-called regular point with (finite) nonzero Lyapunov exponents, then establishing the crucial absolute continuity property, after which one can hope to be able to describe the ergodic properties of the system, to obtain an entropy formula in terms of the Lyapunov exponents, as well as to establish the exponential decay of correlations (see [4, 11, 12] and the references therein). When all Lyapunov exponents are infinite one can still hope to obtain a corresponding structure for a different asymptotic behavior. Here, we consider the construction of stable and unstable invariant manifolds, which already requires a substantial amount of work. Certainly, this is an ambitious project, and one should stress that the theory is only in the beginning. In another direction, besides the polynomial behavior we also consider the impulsive behavior. Taking again billiards as a prototype this would correspond for example to apply a magnetic field at fixed instants of time, thus causing an abrupt change in the dynamics inside the billiard.

2 Nonuniform Polynomial Dichotomies

Let $B(X)$ be the space of bounded linear operators in a Banach space X. We consider the linear impulsive differential equation

$$\begin{aligned} x' &= A(t)x, \quad t \geq 0,\ t \neq \tau_i, \\ x\left(\tau_i^+\right) &= B_i x(\tau_i), \quad i \in \mathbb{N}, \end{aligned} \tag{1}$$

where $I = \{\tau_i\}_{i=1}^{\infty}$ is a sequence of numbers

$$0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots, \quad \lim_{i\to\infty} \tau_i = \infty.$$

We assume that $A(t) \in B(X)$ for each $t \geq 0$, with $t \mapsto A(t)$ at most with discontinuities of the first kind at the points τ_i, and that $B_i \in B(X)$ with $B_i^{-1} \in B(X)$ for $i \in \mathbb{N}$. We also assume that there exist $l > 0$ and $\omega \in \mathbb{N}$ such that each interval of length l contains at most a number ω of elements of I. In particular, taking $\gamma < -1$ and, without loss of generality, letting $l = s + 1$, this implies that

$$\sum_{s\leq\tau_i} (\tau_i + 1)^\gamma \leq \omega(s+1)^\gamma + \omega\big(2(s+1)\big)^\gamma + \cdots = \omega(s+1)^\gamma \lambda_\gamma,$$

where we have set $\lambda_\gamma = \sum_{n=1}^{\infty} n^\gamma$.

Now let $T(t,s)$ be the evolution operator satisfying $T(t,s)x(s) = x(t)$ for any $t, s \geq 0$ and any solution $x(t)$ of (1). Let $\rho\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing function with $\rho(0) = 0$. We say that (1) admits a *ρ-nonuniform exponential dichotomy* if there exist projections $P(t)$ for $t \geq 0$ such that

$$P(t)T(t,s) = T(t,s)P(s), \quad t, s \geq 0, \tag{2}$$

and there exist constants

$$a < 0 \leq b, \quad \varepsilon \geq 0 \quad \text{and} \quad K \geq 1 \tag{3}$$

such that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq K e^{a(\rho(t)-\rho(s))+\varepsilon\rho(s)}, \\ \|T(t,s)^{-1}Q(t)\| &\leq K e^{-b(\rho(t)-\rho(s))+\varepsilon\rho(t)} \end{aligned}$$

for every $t \geq s \geq 0$, where $Q(t) = I - P(t)$ is the complementary projection of $P(t)$. When $\varepsilon = 0$ we say that (1) admits a *(uniform) exponential dichotomy*. We then define the *stable* and *unstable subspaces* for each $t \geq 0$ by

$$E(t) = P(t)(X) \quad \text{and} \quad F(t) = Q(t)(X).$$

The notion of polynomial dichotomy is obtained by taking $\rho(t) = \log(1+t)$. Namely, we say that (1) admits a *nonuniform polynomial dichotomy* if there exist projections $P(t)$ for $t \geq 0$ satisfying (2), and there exists constants as in (3) such that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq K\left(\frac{t+1}{s+1}\right)^a (s+1)^\varepsilon, \\ \|T(t,s)^{-1}Q(t)\| &\leq K\left(\frac{t+1}{s+1}\right)^{-b} (t+1)^\varepsilon \end{aligned} \tag{4}$$

for every $t \geq s \geq 0$. Again, when $\varepsilon = 0$ we say that (1) admits a *(uniform) polynomial dichotomy*.

We note that the above notion of nonuniform polynomial dichotomy is not the one introduced in [7]. However, as also noted there both follow from the existence of negative and positive polynomial Lyapunov exponents (see Theorem 4 below for a rigorous statement for impulsive equations in finite-dimensional spaces). On the other hand, it is easy to verify that they coincide on any interval $(c, +\infty)$ with $c > 0$ sufficiently large, and the construction of stable invariant manifolds only depends on the asymptotic behavior of the dichotomy.

We present an example of nonuniform polynomial dichotomy.

Example 1 Consider the impulsive differential equation

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -\theta/(t+1)+c(t) & 0 \\ 0 & \theta/(t+1)+c(t) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad t \geq 0,\ t \notin \mathbb{N}, \tag{5}$$

$$\begin{pmatrix} z_1(i^+) \\ z_2(i^+) \end{pmatrix} = \begin{pmatrix} (1+1/i)^\rho & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1(i) \\ z_2(i) \end{pmatrix}, \quad i \in \mathbb{N}, \tag{6}$$

where

$$c(t) = \alpha \log(t+1) \cos\log(t+1)/(t+1),$$

and where θ, α and ρ are positive constants. The evolution operator of the nonimpulsive equation is given by

$$\widetilde{T}(t,s) = \begin{pmatrix} \widetilde{U}(t,s) & 0 \\ 0 & \widetilde{V}(t,s) \end{pmatrix},$$

where

$$\widetilde{U}(t,s) = \exp\left[(-\theta+\alpha)d(t)\log\left(\frac{t+1}{1+s}\right)\right],$$

$$\widetilde{V}(t,s) = \exp\left[(\theta+\alpha)d(t)\log\left(\frac{t+1}{1+s}\right)\right],$$

and

$$\begin{aligned} d(t) = \alpha\big[&\log(t+1)\big(\sin\log(t+1)-1\big)+\cos\log(t+1) \\ &-\cos\log(s+1)-\log(s+1)\big(\sin\log(s+1)-1\big)\big]. \end{aligned}$$

Set $P(s)(x,y) = x$ and $Q(s)(x,y) = y$ for $s \geq 0$. We have

$$T(t,s)P(s) = \begin{pmatrix} U(t,s) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{U}(t,s) & 0 \\ 0 & 0 \end{pmatrix}$$

for $i < s \leq t \leq i+1$, and

$$T(t,s)P(s) = \begin{pmatrix} U(t,s) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{U}(t,s)(\frac{i+1}{j})^\rho & 0 \\ 0 & 0 \end{pmatrix}$$

for $j-1 < s \leq j < i < t \leq i+1$. On the other hand,

$$T(t,s)^{-1}Q(t) = \begin{pmatrix} 0 & 0 \\ 0 & V^{-1}(t,s) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{V}^{-1}(t,s) \end{pmatrix}$$

for $s \le t$. Therefore,

$$\begin{aligned}\|T(t,s)P(s)\| &\le e^{2\alpha}\left(\frac{t+s}{s+1}\right)^{-\theta+\alpha}(s+1)^{2\alpha}\left(\frac{i+1}{j}\right)^{\rho}\\ &\le 2^{\rho}e^{2\alpha}\left(\frac{t+1}{s+1}\right)^{-\theta+\alpha+\rho}(s+1)^{2\alpha},\end{aligned}$$

and

$$\|T(t,s)^{-1}Q(t)\| \le e^{2\alpha}\left(\frac{t+1}{s+1}\right)^{-(\theta+\alpha)}(t+1)^{2\alpha}$$

for every $t \ge s > 0$. This implies that the pair of (5)–(6) admits a nonuniform polynomial dichotomy with

$$K = 2^{\rho}e^{2\alpha}, \quad a = -\theta+\alpha+\rho < 0, \quad b = \theta+\alpha, \quad \text{and} \quad \varepsilon = 2\alpha.$$

3 Stable Invariant Manifolds

We establish in this section the existence of stable invariant manifolds under sufficiently small perturbations of a nonuniform polynomial dichotomy. Namely, we consider the nonlinear equation

$$\begin{aligned} x' &= A(t)x + f(t,x), \quad t \ge 0,\ t \ne \tau_i,\\ x(\tau_i^+) &= B_i x(\tau_i) + g_i(x(\tau_i)), \quad i \in \mathbb{N}, \end{aligned} \tag{7}$$

where:

(a) $f\colon \mathbb{R}^+ \times X \to X$ with $f(t,0) = 0$ for every $t \ge 0$, such that $t \mapsto f(t,x)$ has at most discontinuities of the first kind at the points τ_i, and $g_i\colon X \to X$ with $g_i(0) = 0$ for every $i \in \mathbb{N}$;

(b) for some constants $c_1, c_2 > 0$ and $q_1, q_2 > 0$ we have

$$\|f(t,\xi_1) - f(t,\xi_2)\| \le c_1\|\xi_1 - \xi_2\|\left(\|\xi_1\|^{q_1} + \|\xi_2\|^{q_1}\right) \tag{8}$$

and

$$\|g_i(\xi_1) - g_i(\xi_2)\| \le c_2\|\xi_1 - \xi_2\|\left(\|\xi_1\|^{q_2} + \|\xi_2\|^{q_2}\right) \tag{9}$$

for every $t \ge 0$, $i \in \mathbb{N}$, and $\xi_1, \xi_2 \in X$.

Inequalities (8) and (9) (that mimic a related condition in [3]) mean that the perturbations f and g_i are of sufficiently high order. This ensures that they are sufficiently small in a neighborhood of the origin.

The stable manifolds are obtained as graphs of Lipschitz functions. We first describe the class of functions to be considered. For each $s \ge 0$, let $B_s(\varrho) \subset E(s)$ be the open ball of radius ϱ centered at zero, and set

$$\beta = \varepsilon(1 + 2/q) + 1/q, \quad q = \min\{q_1, q_2\}. \tag{10}$$

Given $\delta > 0$, we consider the set of initial conditions

$$Z_\beta = Z_\beta(\delta) = \big\{(s,\xi) : s \geq 0,\ \xi \in B_s\big(\delta(s+1)^{-\beta}\big)\big\},$$

and we denote by X_β the space of functions $\Phi\colon Z_\beta \to X$ that are left-continuous in s, at most with discontinuities of the first kind at the points τ_i, such that

$$\Phi(s,0) = 0, \qquad \Phi\big(s, B_s\big(\delta(s+1)^{-\beta}\big)\big) \subset F(s),$$

and

$$\big\|\Phi(s,\xi_1) - \Phi(s,\xi_2)\big\| \leq \|\xi_1 - \xi_2\| \tag{11}$$

for every $s \geq 0$ and $\xi_1, \xi_2 \in B_s(\delta(s+1)^{-\beta})$. One can easily verify that X_β is a Banach space with the norm

$$|\Phi|' = \sup\left\{\frac{\|\Phi(s,\xi)\|}{\|\xi\|} : s \geq 0 \text{ and } \xi \in B_s\big(\delta(s+1)^{-\beta}\big) \setminus \{0\}\right\}.$$

For each $\Phi \in X_\beta$ we consider the graph

$$\mathcal{W} = \mathcal{W}_\Phi = \big\{\big(s,\xi,\Phi(s,\xi)\big) : (s,\xi) \in Z_\beta\big\}.$$

Moreover, for each $(s, u(s), v(s)) \in \mathbb{R}^+ \times E(s) \times F(s)$ we consider the semiflow

$$\Psi_\kappa\big(s, u(s), v(s)\big) = \big(t, u(t), v(t)\big), \quad \kappa = t - s \geq 0 \tag{12}$$

generated by (7), where

$$\begin{aligned} u(t) &= T(t,s)u(s) + \int_s^t T(t,\tau)P(\tau)f\big(\tau, u(\tau), v(\tau)\big)d\tau \\ &\quad + \sum_{s \leq \tau_i < t} T\big(t, \tau_i^+\big)P\big(\tau_i^+\big)g_i\big(u(\tau_i), v(\tau_i)\big), \end{aligned} \tag{13}$$

$$\begin{aligned} v(t) &= T(t,s)v(s) + \int_s^t T(t,\tau)Q(\tau)f\big(\tau, u(\tau), v(\tau)\big)d\tau \\ &\quad + \sum_{s \leq \tau_i < t} T\big(t, \tau_i^+\big)Q\big(\tau_i^+\big)g_i\big(u(\tau_i), v(\tau_i)\big). \end{aligned} \tag{14}$$

The following is our stable manifold theorem, in the sense that we obtain the unique graph of the form $\mathcal{W}_\Phi$ (for some function $\Phi \in X_\beta$) which is invariant under the semiflow.

Theorem 1 *Assume that* (1) *admits a nonuniform polynomial dichotomy. If* $a + \beta < 0$, *then there exist constants* $\delta > 0$ *and* $\eta > 1$, *and a unique function* $\Phi \in X_\beta$ *such that*

$$\Psi_\kappa\big(s,\xi,\Phi(s,\xi)\big) \in \mathcal{W}_\Phi \quad \textit{for every} \quad (s,\xi) \in Z_{\beta+\varepsilon}(\delta/\eta),\ \kappa \geq 0. \tag{15}$$

Furthermore, *there exists a constant* $d > 0$ *such that*

$$\big\|\Psi_\kappa\big(s,\xi_1,\Phi(s,\xi_1)\big) - \Psi_\kappa\big(s,\xi_2,\Phi(s,\xi_2)\big)\big\| \leq d\left(1 + \frac{\kappa}{s+1}\right)^a (s+1)^\varepsilon \|\xi_1 - \xi_2\| \tag{16}$$

for every $\kappa \geq 0$ *and* $(s,\xi_1), (s,\xi_2) \in Z_{\beta+\varepsilon}(\delta/\eta)$.

Proof Before starting the proof of the theorem we describe the strategy. We first observe that in view of the desired forward invariance of $\mathcal{W}_\Phi$ under the semiflow (see (15)) the solutions in $\mathcal{W}_\Phi$ must be of the form

$$t \mapsto (t, u(t), \Phi(t, u(t))).$$

With this observation in mind, the proof consists essentially of two steps:

(1) We first find a solution $u = u^\Phi$ of (13) for each given $\Phi \in X_\beta$, with $v(t) = \Phi(t, u^\Phi(t))$ (thus ensuring the forward invariance of $\mathcal{W}_\Phi$ under the semiflow). This is the content of Lemma 1. Some auxiliary properties of u^Φ are obtained in Lemmas 3 and 4.
(2) Substituting

$$u(t) = u^\Phi(t) \quad \text{and} \quad v(t) = \Phi(t, u^\Phi(t))$$

in (14), which also needs to be satisfied so that $\mathcal{W}_\Phi$ is invariant, we then show that there is a unique Φ satisfying this identity.

In particular, the solutions in $\mathcal{W}_\Phi$ will have the form

$$t \mapsto (t, u^\Phi(t), \Phi(t, u^\Phi(t))).$$

Moreover, before solving (14) we first show that it is equivalent to another equation (see Lemma 2), which allows us to solve the problem by taking the unique fixed point of a contraction operator in a Banach space (see Lemma 5).

We proceed with the proof of the theorem. Let X_β^* be the space of functions $\Phi\colon \mathbb{R}^+ \times X \to X$ such that $\Phi|_{Z_\beta} \in X_\beta$, and

$$\Phi(s, \xi) = \Phi(s, \delta(s+1)^{-\beta}\xi/\|\xi\|) \quad \text{for } s \geq 0,\ \xi \notin B_s(\delta(s+1)^{-\beta}).$$

There is a one-to-one correspondence between X_β and X_β^*. Moreover, X_β^* is a Banach space with the norm $\Phi \mapsto |\Phi|_{X_\beta}|'$, and one can show that for each $\Phi \in X_\beta^*$ (see [6]),

$$\|\Phi(s, \xi_1) - \Phi(s, \xi_2)\| \leq 2\|\xi_1 - \xi_2\| \quad \text{for } s \geq 0,\ \xi_1, \xi_2 \in E(s). \tag{17}$$

Now let Ω be the space of left-continuous functions $u\colon [s, \infty) \to X$ at most with discontinuities of the first kind at the points τ_i, with $u(s) = \xi$, such that $u(t) \in E(t)$ for every $t \geq s$, and $\|u\|^* \leq \delta(s+1)^{-\beta}$, where

$$\|u\|^* = \frac{1}{2K} \sup\left\{ \frac{\|u(t)\|}{[(t+1)/(s+1)]^a (s+1)^\varepsilon} : t \geq s \right\}.$$

It is easy to show that Ω is a Banach space with the norm $\|\cdot\|^*$.

Step 1. Solution in the Stable Direction

We first obtain a solution u of (13) for each Φ.

Lemma 1 *Given $\delta > 0$ sufficiently small and $(s, \xi) \in Z_\beta$, for each $\Phi \in X_\beta^*$ there exists a unique $u = u_\xi^\Phi \in \Omega$ satisfying (13) for every $t \geq s$. Moveover, there exists $\eta > 1$ such that*

$$\|u^\Phi(t)\| \leq \eta \left(\frac{t+1}{s+1}\right)^a (s+1)^\varepsilon \|\xi\|, \quad t \geq s. \tag{18}$$

Proof Given $(s,\xi)\in Z_\beta$ and $\Phi\in X^*_\beta$, we define an operator L in Ω by

$$(Lu)(t)=T(t,s)\xi+\int_s^t T(t,\tau)P(\tau)f\big(\tau,u(\tau),\Phi\big(\tau,u(\tau)\big)\big)d\tau \\ +\sum_{s\le\tau_i<t}T\big(t,\tau_i^+\big)P\big(\tau_i^+\big)g_i\big(u(\tau_i),\Phi\big(\tau_i,u(\tau_i)\big)\big)$$

for each $t\ge s$. One can verify that Lu is left-continuous in $[s,\infty)$ at most with discontinuities of the first kind at the points τ_i, and that $(Lu)(s)=\xi$ and $(Lu)(t)\in E(t)$ for every $t\ge s$. By the assumptions (a) and (b) and by (17), for each $\tau\ge s$ we have

$$\begin{aligned}a(\tau)&:=\big\|f\big(\tau,u(\tau),\Phi\big(\tau,u(\tau)\big)\big)\big\|\\&\le c_1\big(\|u(\tau)\|+\big\|\Phi\big(\tau,u(\tau)\big)\big\|\big)\big(\|u(\tau)\|+\big\|\Phi\big(\tau,u(\tau)\big)\big\|\big)^{q_1}\\&\le 3^{q_1+1}c_1\|u(\tau)\|^{q_1+1}\\&\le 6^{q_1+1}c_1K^{q_1+1}\left(\frac{\tau+1}{s+1}\right)^{a(q_1+1)}(s+1)^{\varepsilon(q_1+1)}\big(\|u\|^*\big)^{q_1+1}\end{aligned}\tag{19}$$

and similarly, for each $\tau_i\ge s$ we have

$$\begin{aligned}b_i&:=\big\|g_i\big(u(\tau_i),\Phi\big(\tau_i,u(\tau_i)\big)\big)\big\|\\&\le c_2\big(\|u(\tau_i)\|+\big\|\Phi\big(\tau_i,u(\tau_i)\big)\big\|\big)\big(\|u(\tau_i)\|+\big\|\Phi\big(\tau_i,u(\tau_i)\big)\big\|\big)^{q_2}\\&\le 3^{q_2+1}c_2\|u(\tau_i)\|^{q_2+1}\\&\le 6^{q_2+1}c_2K^{q_2+1}\left(\frac{\tau_i+1}{s+1}\right)^{a(q_2+1)}(s+1)^{\varepsilon(q_2+1)}\big(\|u\|^*\big)^{q_2+1}.\end{aligned}\tag{20}$$

It follows from (4), (19) and (20) that

$$\begin{aligned}\big\|(Lu)(t)\big\|&\le\big\|T(t,s)\big\|\cdot\|\xi\|+\int_s^t\big\|T(t,\tau)P(\tau)\big\|a(\tau)d\tau+\sum_{s\le\tau_i<t}\big\|T\big(t,\tau_i^+\big)P\big(\tau_i^+\big)\big\|b_i\\&\le K\left(\frac{t+1}{s+1}\right)^a(s+1)^\varepsilon\|\xi\|\\&\quad+6^{q_1+1}c_1K^{q_1+2}\left(\frac{t+1}{s+1}\right)^a(s+1)^{\varepsilon(q_1+1)-aq_1}\big(\|u\|^*\big)^{q_1+1}\int_s^t(\tau+1)^{\varepsilon+aq_1}d\tau\\&\quad+6^{q_2+1}c_2K^{q_2+2}\left(\frac{t+1}{s+1}\right)^a(s+1)^{\varepsilon(q_2+1)-aq_2}\big(\|u\|^*\big)^{q_2+1}\sum_{s\le\tau_i<t}(\tau_i+1)^{\varepsilon+aq_2}\\&\le K\left(\frac{t+1}{s+1}\right)^a(s+1)^\varepsilon\|\xi\|+\frac{6^{q_1+1}c_1K^{q_1+2}}{|aq_1+\varepsilon+1|}\left(\frac{t+1}{s+1}\right)^a(s+1)^{\varepsilon q_1+2\varepsilon+1}\big(\|u\|^*\big)^{q_1+1}\\&\quad+6^{q_2+1}c_2\omega\lambda_{\varepsilon+aq_2}K^{q_2+2}\left(\frac{t+1}{s+1}\right)^a(s+1)^{\varepsilon q_2+2\varepsilon}\big(\|u\|^*\big)^{q_2+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|Lu\|^* &\le \frac{1}{2}\|\xi\| + \frac{3\cdot 6^{q_1}c_1K^{q_1+1}}{|aq_1+\varepsilon+1|}(s+1)^{\varepsilon q_1+\varepsilon+1}\big(\|u\|^*\big)^{q_1+1}\\ &\quad + 3\cdot 6^{q_2}c_2\omega\lambda_{\varepsilon+aq_2}K^{q_2+1}(s+1)^{\varepsilon q_2+\varepsilon}\big(\|u\|^*\big)^{q_2+1}\\ &\le \frac{1}{2}\left(1+\frac{6^{q_1+1}c_1\delta^{q_1}K^{q_1+1}}{|aq_1+\varepsilon+1|}+6^{q_2+1}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}K^{q_2+1}\right)\delta(s+1)^{-\beta}.\end{aligned}$$

Now we observe that

$$\frac{1}{2}\left(1+\frac{6^{q_1+1}c_1\delta^{q_1}K^{q_1+1}}{|aq_1+\varepsilon+1|}+6^{q_2+1}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}K^{q_2+1}\right)<1$$

for any sufficiently small δ. Thus, provided that δ is sufficiently small we have $L(\Omega)\subset\Omega$. Now we show that L is a contraction. Given $u_1,u_2\in\Omega$, for each $\tau\ge s$ we have

$$\begin{aligned}c(\tau) &:= \big\|f\big(\tau,u_1(\tau),\Phi\big(\tau,u_1(\tau)\big)\big)-f\big(\tau,u_2(\tau),\Phi\big(\tau,u_2(\tau)\big)\big)\big\|\\ &\le 3^{q_1+1}c_1\big\|u_1(\tau)-u_2(\tau)\big\|\big(\big\|u_1(\tau)\big\|^{q_1}+\big\|u_2(\tau)\big\|^{q_1}\big)\\ &\le 6^{q_1+1}c_1\delta^{q_1}K^{q_1}\left(\frac{\tau+1}{s+1}\right)^{aq_1}(s+1)^{\varepsilon q_1-\beta q_1}\big\|u_1(\tau)-u_2(\tau)\big\|\\ &\le 2\cdot 6^{q_1+1}c_1\delta^{q_1}K^{q_1+1}\left(\frac{\tau+1}{s+1}\right)^{aq_1+a}(s+1)^{\varepsilon(q_1+1)-\beta q_1}\|u_1-u_2\|^* \end{aligned}\tag{21}$$

and similarly, for each $\tau_i\ge s$ we have

$$\begin{aligned}d_i &:= \big\|g_i\big(u_1(\tau_i),\Phi\big(\tau_i,u_1(\tau_i)\big)\big)-g_i\big(u_2(\tau_i),\Phi\big(\tau_i,u_2(\tau_i)\big)\big)\big\|\\ &\le 3^{q_2+1}c_2\big\|u_1(\tau_i)-u_2(\tau_i)\big\|\big(\big\|u_1(\tau_i)\big\|^{q_2}+\big\|u_2(\tau_i)\big\|^{q_2}\big)\\ &\le 6^{q_2+1}c_2\delta^{q_2}K^{q_2}\left(\frac{\tau_i+1}{s+1}\right)^{aq_2}(s+1)^{\varepsilon q_2-\beta q_2}\|u_1(\tau_i)-u_2(\tau_i)\|\\ &\le 2\cdot 6^{q_2+1}c_2\delta^{q_2}K^{q_2+1}\left(\frac{\tau_i+1}{s+1}\right)^{aq_2+a}(s+1)^{\varepsilon(q_2+1)-\beta q_2}\|u_1-u_2\|^*. \end{aligned}\tag{22}$$

It follows from (4), (21) and (22) that

$$\begin{aligned}&\|(Lu_1)(t)-(Lu_2)(t)\|\\ &\le \int_s^t\|T(t,\tau)P(\tau)\|c(\tau)\,d\tau+\sum_{s\le\tau_i<t}\|T\big(t,\tau_i^+\big)P\big(\tau_i^+\big)\|d_i\\ &\le 2\cdot 6^{q_1+1}c_1\delta^{q_1}K^{q_1+2}\left(\frac{t+1}{s+1}\right)^{a}(s+1)^{\alpha_1}\|u_1-u_2\|^*\int_s^t(\tau+1)^{\varepsilon+aq_1}\,d\tau\\ &\quad +2\cdot 6^{q_2+1}c_2\delta^{q_2}K^{q_2+2}\left(\frac{t+1}{s+1}\right)^{a}(s+1)^{\alpha_2}\|u_1-u_2\|^*\sum_{s\le\tau_i<t}(\tau_i+1)^{\varepsilon+aq_2}\\ &\le \frac{2\cdot 6^{q_1+1}c_1\delta^{q_1}K^{q_1+2}}{|aq_1+\varepsilon+1|}\left(\frac{t+1}{s+1}\right)^{a}(s+1)^{\varepsilon q_1+2\varepsilon+1-\beta q_1}\|u_1-u_2\|^*\\ &\quad +2\cdot 6^{q_2+1}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}K^{q_2+2}\left(\frac{t+1}{s+1}\right)^{a}(s+1)^{\varepsilon q_2+2\varepsilon-\beta q_2}\|u_1-u_2\|^*\end{aligned}$$

where

$$\alpha_i = \varepsilon(q_i + 1) - \beta q_i - a q_i \tag{23}$$

for $i = 1, 2$. Taking δ sufficiently small such that

$$\vartheta_1 = \frac{6^{q_1+1} c_1 \delta^{q_1} K^{q_1+1}}{|aq_1 + \varepsilon + 1|} + 6^{q_2+1} c_2 \omega \lambda_{\varepsilon + a q_2} \delta^{q_2} K^{q_2+1} < 1,$$

we obtain

$$\|Lu_1 - Lu_2\|^* \leq \vartheta_1 \|u_1 - u_2\|^*.$$

Therefore, L is a contraction, and there exists a unique $u = u^\Phi \in \Omega$ such that $Lu = u$. Moveover, it is easy to show that

$$\|u\|^* \leq \frac{1}{2}\|\xi\| + \vartheta_1 \|u\|^*,$$

which yields (18) with $\eta = K/(1 - \vartheta_1)$. □

Step 2. Equivalent Problem

Now we rewrite (13) in an equivalent form. Let $u = u_\xi^\Phi$ be the unique function given by Lemma 1.

Lemma 2 *Given $\delta > 0$ sufficiently small and $\Phi \in X_\beta^*$, the following properties hold*:

1. *for each $(s, \xi) \in Z_\beta$ and $t \geq s$, if*

$$\Phi\big(t, u(t)\big) = T(t,s)\Phi(s,\xi) + \int_s^t T(t,\tau)Q(\tau) f\big(\tau, u(\tau), \Phi\big(\tau, u(\tau)\big)\big) d\tau + \sum_{s \leq \tau_i < t} T\big(t, \tau_i^+\big) Q\big(\tau_i^+\big) g_i\big(u(\tau_i), \Phi\big(\tau_i, u(\tau_i)\big)\big), \tag{24}$$

then

$$\Phi(s,\xi) = -\int_s^\infty T(\tau,s)^{-1} Q(\tau) f\big(\tau, u(\tau), \Phi\big(\tau, u(\tau)\big)\big) d\tau - \sum_{s \leq \tau_i} T\big(\tau_i^+, s\big)^{-1} Q\big(\tau_i^+\big) g_i\big(u(\tau_i), \Phi\big(\tau_i, u(\tau_i)\big)\big). \tag{25}$$

2. *if identity (25) holds for each $s \geq 0$ and $\xi \in B_s(\delta(s+1)^{-\beta})$, then (24) holds for each $\xi \in B_s((\delta/\eta)(s+1)^{-(\beta+\varepsilon)})$ and $t \geq s$.*

Proof By the assumption (b) and by (4), (17) and (18), for each $\tau \geq s$ we have

$$A(\tau) := \big\|T(\tau,s)^{-1} Q(\tau)\big\| \cdot \big\| f\big(\tau, u(\tau), \Phi\big(\tau, u(\tau)\big)\big)\big\| \leq 3^{q_1+1} c_1 K \left(\frac{\tau+1}{s+1}\right)^{-b} (\tau+1)^\varepsilon \big\|u(\tau)\big\|^{q_1+1}$$

$$\le 3^{q_1+1}c_1\eta^{q_1+1}K\left(\frac{\tau+1}{s+1}\right)^{-b+a(q_1+1)}(\tau+1)^{\varepsilon}(s+1)^{\varepsilon(q_1+1)}\|\xi\|^{q_1+1}$$

$$\le 3^{q_1+1}c_1\eta^{q_1+1}\delta^{q_1+1}K\left(\frac{\tau+1}{s+1}\right)^{-b+a(q_1+1)}(\tau+1)^{\varepsilon}(s+1)^{\varepsilon(q_1+1)-\beta(q_1+1)}$$

and similarly, for each $\tau_i \ge s$ we have

$$\begin{aligned}
B_i &:= \left\|T\left(\tau_i^+, s\right)^{-1}Q\left(\tau_i^+\right)\right\| \cdot \left\|g_i\left(u(\tau_i), \Phi\left(\tau_i, u(\tau_i)\right)\right)\right\| \\
&\le 3^{q_2+1}c_2K\left(\frac{\tau_i+1}{s+1}\right)^{-b}(\tau_i+1)^{\varepsilon}\left\|u(\tau_i)\right\|^{q_2+1} \\
&\le 3^{q_2+1}c_2\eta^{q_2+1}K\left(\frac{\tau_i+1}{s+1}\right)^{-b+a(q_2+1)}(\tau_i+1)^{\varepsilon}(s+1)^{\varepsilon(q_2+1)}\|\xi\|^{q_2+1} \\
&\le 3^{q_2+1}c_2\eta^{q_2+1}\delta^{q_2+1}K\left(\frac{\tau_i+1}{s+1}\right)^{-b+a(q_2+1)}(\tau_i+1)^{\varepsilon}(s+1)^{\varepsilon(q_2+1)-\beta(q_2+1)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_s^\infty A(\tau)\,d\tau + \sum_{s\le\tau_i} B_i \\
&\le 3^{q_1+1}c_1\eta^{q_1+1}\delta^{q_1+1}K(s+1)^{b-a(q_1+1)+\varepsilon(q_1+1)-\beta(q_1+1)}\int_s^\infty(\tau+1)^{-b+a(q_1+1)+\varepsilon}\,d\tau \\
&\quad + 3^{q_2+1}c_2\eta^{q_2+1}\delta^{q_2+1}K(s+1)^{b-a(q_2+1)+\varepsilon(q_2+1)-\beta(q_2+1)}\sum_{s\le\tau_i}(\tau_i+1)^{-b+a(q_2+1)+\varepsilon} \\
&\le \frac{3^{q_1+1}c_1\eta^{q_1+1}\delta^{q_1+1}K}{|-b+a(q_1+1)+\varepsilon+1|} + 3^{q_2+1}c_2\omega\lambda_{-b+a(q_2+1)+\varepsilon}\eta^{q_2+1}\delta^{q_2+1}K < \infty.
\end{aligned}$$

This implies that the right-hand side of (25) is always well-defined.

Now we assume that (24) holds for each $(s,\xi) \in Z_\beta$ and $t \ge s$. Identity (24) can be written in the form

$$\begin{aligned}
\Phi(s,\xi) = T(t,s)^{-1}\Phi\left(t,u(t)\right) &- \int_s^t T(\tau,s)^{-1}Q(\tau)f\left(\tau,u(\tau),\Phi\left(\tau,u(\tau)\right)\right)d\tau \\
&- \sum_{s\le\tau_i<t} T\left(\tau_i^+,s\right)^{-1}Q\left(\tau_i^+\right)g_i\left(u(\tau_i),\Phi\left(\tau_i,u(\tau_i)\right)\right). \qquad (26)
\end{aligned}$$

By (4), (17) and (18), we obtain

$$\begin{aligned}
&\left\|T(t,s)^{-1}\Phi\left(t,u(t)\right)\right\| \\
&\le 2\delta K\eta\left(\frac{t+1}{s+1}\right)^{-b}(t+1)^{\varepsilon}\left(\frac{t+1}{s+1}\right)^{a}(s+1)^{\varepsilon}(s+1)^{-\beta} \\
&\le 2\delta K\eta(t+1)^{a-b+\varepsilon}(s+1)^{-a+b+\varepsilon-\beta}. \qquad (27)
\end{aligned}$$

We note that since $a-b+\varepsilon<0$ (recall that $a+\varepsilon<a+\beta<0$), the right-hand side of (27) tends to zero when $t\to+\infty$. Therefore, letting $t\to+\infty$ in (26) yields identity (25).

In the other direction, we assume that identity (25) holds for each $s \geq 0$ and $\xi \in B_s(\delta(s+1)^{-\beta})$. For each $\xi \in B_s((\delta/\eta)(s+1)^{-(\beta+\varepsilon)})$ we have

$$\|u(t)\| \leq \eta\left(\frac{t+1}{s+1}\right)^a (s+1)^\varepsilon \|\xi\| \leq \delta(t+1)^{-\beta}\left(\frac{t+1}{s+1}\right)^{a+\beta} \leq \delta(t+1)^{-\beta},$$

and hence, $(t, u(t)) \in Z_\beta$ for any $t \geq s$. It follows from (25) that

$$\begin{aligned} T(t,s)\Phi(s,\xi) = &-\int_s^t T(t,\tau)Q(\tau)f\big(\tau,u(\tau),\Phi\big(\tau,u(\tau)\big)\big)d\tau \\ &-\int_t^\infty T(t,\tau)Q(\tau)f\big(\tau,u(\tau),\Phi\big(\tau,u(\tau)\big)\big)d\tau \\ &-\sum_{s\leq\tau_i<t} T\big(t,\tau_i^+\big)Q\big(\tau_i^+\big)g_i\big(u(\tau_i),\Phi\big(\tau_i,u(\tau_i)\big)\big) \\ &-\sum_{t\leq\tau_i} T\big(t,\tau_i^+\big)Q\big(\tau_i^+\big)g_i\big(u(\tau_i),\Phi\big(\tau_i,u(\tau_i)\big)\big) \\ = &-\int_s^t T(t,\tau)Q(\tau)f\big(\tau,u(\tau),\Phi\big(\tau,u(\tau)\big)\big)d\tau \\ &-\sum_{s\leq\tau_i<t} T\big(t,\tau_i^+\big)Q\big(\tau_i^+\big)g_i\big(u(\tau_i),\Phi\big(\tau_i,u(\tau_i)\big)\big) + \Phi\big(t,u(t)\big), \end{aligned}$$

using (25) in the last identity with (s, ξ) replaced by $(t, u(t))$. □

Step 3. Auxiliary Properties

We obtain here some auxiliary properties for the function u^Φ.

Lemma 3 *Given $\delta > 0$ sufficiently small, there exists $K_1 > 0$ such that*

$$\left\|u_{\xi_1}^\Phi(t) - u_{\xi_2}^\Phi(t)\right\| \leq K_1\left(\frac{t+1}{s+1}\right)^a (s+1)^\varepsilon \|\xi_1 - \xi_2\|, \quad t \geq s \tag{28}$$

for every $\Phi \in X_\beta^*$ and $(s, \xi_1), (s, \xi_2) \in Z_\beta$.

Proof Write $u_i = u_{\xi_i}^\Phi$. By (4), (17) and (18), we have

$$\begin{aligned} C := &\int_s^t \|T(t,\tau)P(\tau)\| \cdot \big\| f\big(\tau,u_1(\tau),\Phi\big(\tau,u_1(\tau)\big)\big) \\ &- f\big(\tau,u_2(\tau),\Phi\big(\tau,u_2(\tau)\big)\big)\big\| d\tau \\ \leq\ & 3^{q_1+1}c_1 K \int_s^t \left(\frac{t+1}{\tau+1}\right)^a (\tau+1)^\varepsilon \big\|u_1(\tau)-u_2(\tau)\big\| \big(\big\|u_1(\tau)\big\|^{q_1} + \big\|u_2(\tau)\big\|^{q_1}\big)d\tau \\ \leq\ & \frac{4\cdot 3^{q_1+1}c_1\delta^{q_1}\eta^{q_1}K^2}{|aq_1+\varepsilon+1|}\left(\frac{t+1}{s+1}\right)^a (s+1)^\varepsilon \|u_1-u_2\|^*, \end{aligned}$$

and

$$\begin{aligned} D &:= \sum_{s \le \tau_i < t} \left\| T\left(t, \tau_i^+\right) P\left(\tau_i^+\right) \right\| \cdot \left\| g_i\left(u_1(\tau_i), \Phi\left(\tau_i, u_1(\tau_i)\right)\right) \right. \\ &\quad \left. - g_i\left(u_2(\tau_i), \Phi\left(\tau_i, u_2(\tau_i)\right)\right) \right\| \\ &\le 3^{q_2+1} c_2 K \sum_{s \le \tau_i < t} \left(\frac{t+1}{\tau_i+1} \right)^a (\tau_i+1)^\varepsilon \\ &\quad \times \left\| u_1(\tau_i) - u_2(\tau_i) \right\| \left(\left\| u_1(\tau_i) \right\|^{q_2} + \left\| u_2(\tau_i) \right\|^{q_2} \right) \\ &\le 4 \cdot 3^{q_2+1} c_2 \omega \lambda_{\varepsilon + a q_2} \delta^{q_2} \eta^{q_2} K^2 \left(\frac{t+1}{s+1} \right)^a (s+1)^\varepsilon \|u_1 - u_2\|^*. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| u_1(t) - u_2(t) \right\| &\le \|T(t,s)\| \cdot \|\xi_1 - \xi_2\| + C + D \\ &\le K \left(\frac{t+1}{s+1} \right)^a (s+1)^\varepsilon \|\xi_1 - \xi_2\| \\ &\quad + \frac{4 \cdot 3^{q_1+1} c_1 \delta^{q_1} \eta^{q_1} K^2}{|a q_1 + \varepsilon + 1|} \left(\frac{t+1}{s+1} \right)^a (s+1)^\varepsilon \|u_1 - u_2\|^* \\ &\quad + 4 \cdot 3^{q_2+1} c_2 \omega \lambda_{\varepsilon + a q_2} \delta^{q_2} \eta^{q_2} K^2 \left(\frac{t+1}{s+1} \right)^a (s+1)^\varepsilon \|u_1 - u_2\|^*. \end{aligned}$$

Setting

$$\vartheta_2 = \frac{2 \cdot 3^{q_1+1} c_1 \delta^{q_1} \eta^{q_1} K}{|a q_1 + \varepsilon + 1|} + 2 \cdot 3^{q_2+1} c_2 \omega \lambda_{\varepsilon + a q_2} \delta^{q_2} \eta^{q_2} K,$$

we obtain

$$\|u_1 - u_2\|^* \le \frac{1}{2} \|\xi_1 - \xi_2\| + \vartheta_2 \|u_1 - u_2\|^*,$$

which yields (28) with $K_1 = K/(1 - \vartheta_2)$. □

Lemma 4 *Given $\delta > 0$ sufficiently small, there exists $K_2 > 0$ such that*

$$\left\| u_\xi^{\Phi_1}(t) - u_\xi^{\Phi_2}(t) \right\| \le K_2 \left(\frac{t+1}{s+1} \right)^a \|\xi\| \cdot |\Phi_1 - \Phi_2|', \quad t \ge s \tag{29}$$

for every $\Phi_1, \Phi_2 \in X_\beta^*$ and $(s, \xi) \in Z_\beta$.

Proof Write $u_i = u_\xi^{\Phi_i}$ for $i = 1, 2$. We first note that

$$\begin{aligned} &\left\| \Phi_1\left(\tau, u_1(\tau)\right) - \Phi_2\left(\tau, u_2(\tau)\right) \right\| \\ &\quad \le \left\| \Phi_1\left(\tau, u_1(\tau)\right) - \Phi_2\left(\tau, u_1(\tau)\right) \right\| + \left\| \Phi_2\left(\tau, u_1(\tau)\right) - \Phi_2\left(\tau, u_2(\tau)\right) \right\| \\ &\quad \le \left\| u_1(\tau) \right\| \cdot |\Phi_1 - \Phi_2|' + 2 \left\| u_1(\tau) - u_2(\tau) \right\|. \end{aligned}$$

With similar arguments to those in Lemmas 1 and 3, we obtain

$$\int_s^t \big\|T(t,\tau)P(\tau)\big\| \cdot \big\|f\big(\tau,u_1(\tau),\Phi_1\big(\tau,u_1(\tau)\big)\big)-f\big(\tau,u_2(\tau),\Phi_2\big(\tau,u_2(\tau)\big)\big)\big\|d\tau$$
$$\le 3^{q_1}c_1K\int_s^t\left(\frac{t+1}{\tau+1}\right)^a(\tau+1)^\varepsilon\big[3\big(\big\|u_1(\tau)-u_2(\tau)\big\|\big)\big(\big\|u_1(\tau)\big\|^{q_1}+\big\|u_2(\tau)\big\|^{q_1}\big)$$
$$+\big(\big\|u_1(\tau)\big\|\cdot|\Phi_1-\Phi_2|'\big)\big(\big\|u_1(\tau)\big\|^{q_1}+\big\|u_2(\tau)\big\|^{q_1}\big)\big]d\tau$$
$$\le 4\cdot 3^{q_1+1}c_1\delta^{q_1}\eta^{q_1}K^2\|u_1-u_2\|^*\int_s^t\left(\frac{t+1}{s+1}\right)^a(\tau+1)^{aq_1+\varepsilon}(s+1)^{\alpha_1}\,d\tau$$
$$+2\cdot 3^{q_1}c_1\delta^{q_1}\eta^{q_1+1}K\|\xi\|\cdot|\Phi_1-\Phi_2|'\int_s^t\left(\frac{t+1}{s+1}\right)^a(\tau+1)^{aq_1+\varepsilon}(s+1)^{\alpha_1}\,d\tau$$
$$\le \frac{2\cdot 3^{q_1}c_1\delta^{q_1}\eta^{q_1}K}{|aq_1+\varepsilon+1|}\big(6K\|u_1-u_2\|^*+\eta\|\xi\|\cdot|\Phi_1-\Phi_2|'\big)\left(\frac{t+1}{s+1}\right)^a,$$

and

$$\sum_{s\le\tau_i<t}\big\|T\big(t,\tau_i^+\big)P\big(\tau_i^+\big)\big\|$$
$$\times\big\|g_i\big(u_1(\tau_i),\Phi_1\big(\tau_i,u_1(\tau_i)\big)\big)-g_i\big(u_2(\tau_i),\Phi_2\big(\tau_i,u_2(\tau_i)\big)\big)\big\|$$
$$\le 3^{q_2}c_2K\sum_{s\le\tau_i<t}\left(\frac{t+1}{\tau_i+1}\right)^a(\tau_i+1)^\varepsilon$$
$$\times\big[3\big(\big\|u_1(\tau_i)-u_2(\tau_i)\big\|\big)\big(\big\|u_1(\tau_i)\big\|^{q_2}+\big\|u_2(\tau_i)\big\|^{q_2}\big)$$
$$+\big(\big\|u_1(\tau_i)\big\|\cdot|\Phi_1-\Phi_2|'\big)\big(\big\|u_1(\tau_i)\big\|^{q_2}+\big\|u_2(\tau_i)\big\|^{q_2}\big)\big]$$
$$\le 4\cdot 3^{q_2+1}c_2\delta^{q_2}\eta^{q_2}K^2\|u_1-u_2\|^*\sum_{s\le\tau_i<t}\left(\frac{t+1}{s+1}\right)^a(\tau_i+1)^{aq_2+\varepsilon}(s+1)^{\alpha_2}$$
$$+2\cdot 3^{q_2}c_2\delta^{q_2}\eta^{q_2+1}K\|\xi\|\cdot|\Phi_1-\Phi_2|'\sum_{s\le\tau_i<t}\left(\frac{t+1}{s+1}\right)^a(\tau_i+1)^{aq_2+\varepsilon}(s+1)^{\alpha_2}$$
$$\le 2\cdot 3^{q_2}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}\eta^{q_2}K\big(6K\|u_1-u_2\|^*+\eta\|\xi\|\cdot|\Phi_1-\Phi_2|'\big)\left(\frac{t+1}{s+1}\right)^a,$$

with α_i for $i=1,2$ as in (23). Summing the two right-hand sides we obtain

$$\big\|u_1(t)-u_2(t)\big\|\left(\frac{t+1}{s+1}\right)^{-a}$$
$$\le\left(\frac{4\cdot 3^{q_1+1}c_1\delta^{q_1}\eta^{q_1}K^2}{|aq_1+\varepsilon+1|}+4\cdot 3^{q_2+1}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}\eta^{q_2}K^2\right)\|u_1-u_2\|^*$$
$$+\left(\frac{2\cdot 3^{q_1}c_1\delta^{q_1}\eta^{q_1+1}K}{|aq_1+\varepsilon+1|}+2\cdot 3^{q_2}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}\eta^{q_2+1}K\right)\|\xi\|\cdot|\Phi_1-\Phi_2|',$$

and thus,

$$\begin{aligned}
&2K\|u_1-u_2\|^* \\
&\quad\le \left(\frac{4\cdot 3^{q_1+1}c_1\delta^{q_1}\eta^{q_1}K^2}{|aq_1+\varepsilon+1|}+4\cdot 3^{q_2+1}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}\eta^{q_2}K^2\right)\|u_1-u_2\|^* \\
&\qquad+\left(\frac{2\cdot 3^{q_1}c_1\delta^{q_1}\eta^{q_1+1}K}{|aq_1+\varepsilon+1|}+2\cdot 3^{q_2}c_2\omega\lambda_{\varepsilon+aq_2}\delta^{q_2}\eta^{q_2+1}K\right)\frac{\|\xi\|\cdot|\Phi_1-\Phi_2|'}{(s+1)^{\varepsilon}}.
\end{aligned}$$

This yields (29), provided that δ is sufficiently small. □

Step 4. Existence and Uniqueness of the Stable Manifold

Now we transform (25) into a fixed point problem, and we show that it has a unique solution Φ in X_β.

Lemma 5 *Given $\delta>0$ sufficiently small, there exists a unique function $\Phi\in X_\beta$ such that* (25) *holds for every $(s,\xi)\in Z_\beta$.*

Proof For each $\Phi\in X_\beta$ and $(s,\xi)\in Z_\beta$, we define an operator J by

$$\begin{aligned}
(J\Phi)(s,\xi)=&-\int_s^\infty T(\tau,s)^{-1}Q(\tau)f\big(\tau,u(\tau),\Phi(\tau,u(\tau))\big)d\tau \\
&-\sum_{s\le\tau_i}T\big(\tau_i^+,s\big)^{-1}Q\big(\tau_i^+\big)g_i\big(u(\tau_i),\Phi(\tau_i,u(\tau_i))\big),
\end{aligned}$$

where $u=u_\xi^\Phi$ is the unique function given by Lemma 1. One can verify that $J\Phi$ is left-continuous in s at most with discontinuities of the first kind at the points τ_i, and that $(J\Phi)(s,0)=0$ for $s\ge 0$. Moreover, writing $u_i=u_{\xi_i}^\Phi$ for $i=1,2$, by (4), (18) and (28), for each $\xi_1,\xi_2\in B_s(\delta(s+1)^{-\beta})$ we have

$$\begin{aligned}
&\int_s^\infty \big\|T(\tau,s)^{-1}Q(\tau)\big\|\cdot\big\|f\big(\tau,u_1(\tau),\Phi(\tau,u_1(\tau))\big) \\
&\qquad-f\big(\tau,u_2(\tau),\Phi(\tau,u_2(\tau))\big)\big\|\,d\tau \\
&\quad\le 2\cdot 3^{q_1+1}c_1\delta^{q_1}\eta^{q_1}KK_1 \\
&\qquad\times\|\xi_1-\xi_2\|\int_s^\infty\left(\frac{\tau+1}{s+1}\right)^{-b+a(q_1+1)}(\tau+1)^{\varepsilon}(s+1)^{\varepsilon(q_1+1)-\beta q_1}\,d\tau \\
&\quad\le\frac{2\cdot 3^{q_1+1}c_1\delta^{q_1}\eta^{q_1}KK_1}{|-b+a(q_1+1)+\varepsilon+1|}\|\xi_1-\xi_2\|,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{s\le\tau_i}\big\|T\big(\tau_i^+,s\big)^{-1}Q\big(\tau_i^+\big)\big\|\cdot\big\|g_i\big(u_1(\tau_i),\Phi(\tau_i,u_1(\tau_i))\big)-g_i\big(u_2(\tau_i),\Phi(\tau_i,u_2(\tau_i))\big)\big\| \\
&\quad\le 2\cdot 3^{q_2+1}c_2\delta^{q_2}\eta^{q_2}KK_1
\end{aligned}$$

$$\times \|\xi_1 - \xi_2\| \sum_{s \le \tau_i} \left(\frac{\tau_i + 1}{s+1} \right)^{-b+a(q_2+1)} (\tau_i + 1)^{\varepsilon} (s+1)^{\varepsilon(q_2+1)-\beta q_2}$$
$$\le 2 \cdot 3^{q_2+1} c_2 \omega \lambda_{-b+a(q_2+1)+\varepsilon} \delta^{q_2} \eta^{q_2} K K_1 \|\xi_1 - \xi_2\|.$$

Therefore, summing the two right-hand sides, we obtain

$$\left\| (J\Phi)(s, \xi_1) - (J\Phi)(s, \xi_2) \right\|$$
$$\le \left(\frac{2 \cdot 3^{q_1+1} c_1 \delta^{q_1} \eta^{q_1} K K_1}{|-b + a(q_1+1) + \varepsilon + 1|} + 2 \cdot 3^{q_2+1} c_2 \omega \lambda_{-b+a(q_2+1)+\varepsilon} \delta^{q_2} \eta^{q_2} K K_1 \right) \|\xi_1 - \xi_2\|.$$

Provided that δ is sufficiently small, we obtain

$$\left\| (J\Phi)(s, \xi_1) - (J\Phi)(s, \xi_2) \right\| \le \|\xi_1 - \xi_2\|,$$

and hence, $J(X_\beta) \subset X_\beta$. Now we show that J is a contraction. Given $\Phi_1, \Phi_2 \in X_\beta$ and writing $u_i = u_\xi^{\Phi_i}$ for $i = 1, 2$, by (4), (17), (18) and (29), for each $(s, \xi) \in Z_\beta$ we have

$$\int_s^\infty \left\| T(\tau, s)^{-1} Q(\tau) \right\| \cdot \left\| f\big(\tau, u_1(\tau), \Phi_1(\tau, u_1(\tau))\big) - f\big(\tau, u_2(\tau), \Phi_2(\tau, u_2(\tau))\big) \right\| d\tau$$
$$\le 2 \cdot 3^{q_1} c_1 \eta^{q_1} \delta^{q_1} K(\eta + 3K_2) \|\xi\| \cdot |\Phi_1 - \Phi_2|'$$
$$\times \int_s^\infty \left(\frac{\tau + 1}{s+1} \right)^{-b+a(q_1+1)} (\tau + 1)^{\varepsilon} (s+1)^{q_1(\varepsilon-\beta)+\varepsilon} \, d\tau$$
$$\le \frac{2 \cdot 3^{q_1} c_1 \eta^{q_1} \delta^{q_1} K(\eta + 3K_2)}{|-b + a(q_1+1) + \varepsilon + 1|} \|\xi\| \cdot |\Phi_1 - \Phi_2|',$$

and

$$\sum_{s \le \tau_i} \left\| T\big(\tau_i^+, s\big)^{-1} Q\big(\tau_i^+\big) \right\| \cdot \left\| g_i\big(u_1(\tau_i), \Phi_1\big(\tau_i, u_1(\tau_i)\big)\big) - g_i\big(u_2(\tau_i), \Phi_2\big(\tau_i, u_2(\tau_i)\big)\big) \right\|$$
$$\le 2 \cdot 3^{q_2} c_2 \eta^{q_2} \delta^{q_2} K(\eta + 3K_2) \|\xi\| \cdot |\Phi_1 - \Phi_2|'$$
$$\times \sum_{s \le \tau_i} \left(\frac{\tau_i + 1}{s+1} \right)^{-b+a(q_2+1)} (\tau_i + 1)^{\varepsilon} (s+1)^{q_2(\varepsilon-\beta)+\varepsilon}$$
$$\le 2 \cdot 3^{q_2} c_2 \omega \lambda_{-b+a(q_2+1)+\varepsilon} \eta^{q_2} \delta^{q_2} K(\eta + 3K_2) \|\xi\| \cdot |\Phi_1 - \Phi_2|'.$$

Therefore, summing the two right-hand sides,

$$\left\| (J\Phi_1)(s, \xi) - (J\Phi_2)(s, \xi) \right\|$$
$$\le \left(\frac{2 \cdot 3^{q_1} c_1 \eta^{q_1} \delta^{q_1} K(\eta + 3K_2)}{|-b + a(q_1+1) + \varepsilon + 1|} \right.$$
$$\left. + 2 \cdot 3^{q_2} c_2 \omega \lambda_{-b+a(q_2+1)+\varepsilon} \eta^{q_2} \delta^{q_2} K(\eta + 3K_2) \right) \|\xi\| \cdot |\Phi_1 - \Phi_2|',$$

and provided that δ is sufficiently small, the operator J is a contraction. Therefore, there exists a unique function $\Phi \in X_\beta$ such that (25) holds for every $(s, \xi) \in Z_\beta$. □

We can now complete the proof of Theorem 1. By Lemma 1, for each $(s,\xi)\in Z_\beta$ and $\Phi\in X_\beta$ there exists a unique function $u=u_\xi^\Phi\in\Omega$. By Lemmas 2 and 5, for each $s\ge 0$ and $\xi\in B_s((\delta/\eta)(s+1)^{-(\beta+\varepsilon)})$ there exists a unique function $\Phi\in X_\beta$ such that (24) holds. According to the strategy described in the beginning of the proof, this precisely shows that (15) holds (for any sufficiently small δ). For each $\kappa, s\ge 0$, $\xi_1,\xi_2\in B((\delta/\eta)(s+1)^{-(\beta+\varepsilon)})$ and $t\ge\xi$, by Lemma 3 we have

$$\begin{aligned}&\big\|\Psi_{t-s}\big(s,\xi_1,\Phi(s,\xi_1)\big)-\Psi_{t-s}\big(s,\xi_2,\Phi(s,\xi_2)\big)\big\|\\&\quad=\big\|\big(t,u_1(t),\Phi\big(t,u_1(t)\big)\big)-\big(t,u_2(t),\Phi\big(t,u_2(t)\big)\big)\big\|\\&\quad\le 2\big\|u_1(t)-u_2(t)\big\|\le 2K_1\left(1+\frac{t-s}{s+1}\right)^a(s+1)^\varepsilon\|\xi_1-\xi_2\|,\end{aligned}$$

where $u_i=u_{\xi_i}^\Phi$ for $i=1,2$. This completes the proof of the theorem. □

As we already mentioned, the statement of Theorem 1 can indeed be considered a stable manifold theorem. We believe that by considering perturbations f and g_i of class C^1 (although we emphasize that we do not need this assumption in Theorem 1), one can show that $\mathcal{W}_\Phi$ is also of class C^1, using for example techniques in [8], although the details should be quite elaborate.

4 Further Developments

We describe in this section several consequences of the stable manifold theorem (Theorem 1). In particular, we consider the case of nonuniform polynomial contractions, and we establish the existence of unstable invariant manifolds.

4.1 Lyapunov Stability

We first consider the case when there exists only contraction. Let again $T(t,s)$ be the evolution operator of the linear impulsive equation (1). We say that the equation admits a *nonuniform polynomial contraction* if there exist constants $a<0$, $\varepsilon\ge 0$ and $K\ge 1$ such that

$$\big\|T(t,s)\big\|\le K\left(\frac{t+1}{s+1}\right)^a(s+1)^\varepsilon$$

for every $t\ge s\ge 0$. This corresponds to taking $P(t)=\mathrm{Id}$ for each $t\ge 0$ in the notion of nonuniform polynomial dichotomy.

We also consider the nonlinear equation (7) for some functions f and g_i satisfying conditions (a) and (b) in Sect. 3. The following is a particular case of the stable manifold theorem (Theorem 1).

Theorem 2 *Assume that* (1) *admits a nonuniform polynomial contraction. If $a+\beta<0$, then there exist constants $\delta, d>0$ such that any solutions $x_1(t)$ and $x_2(t)$ of* (7) *satisfy*

$$\big\|x_1(t)-x_2(t)\big\|\le d\left(\frac{t+1}{s+1}\right)^a(s+1)^\varepsilon\big\|x_1(s)-x_2(s)\big\|$$

for every $t\ge s\ge 0$, whenever $\|x_1(s)\|, \|x_2(s)\|<\delta(s+1)^{-\beta}$.

The proof of Theorem 2 essentially reduces to the proofs of Lemmas 1 and 3.

4.2 Unstable Invariant Manifolds

We describe in this section the construction of unstable invariant manifolds. In this case we consider (1) and (7) in $\mathbb{R}^-$, and a corresponding notion of nonuniform polynomial dichotomy. Namely, we say that (1) admits a nonuniform polynomial dichotomy in $\mathbb{R}^-$ if there exist projections $P(t)$ for $t \leq 0$ such that

$$P(t)T(t,s) = T(t,s)P(s), \quad t,s \leq 0,$$

and there exist constants $a \leq 0 < b$, $\varepsilon \geq 0$ and $K \geq 1$ such that

$$\left\|T(t,s)P(s)\right\| \leq K\left(\frac{|t|+1}{|s|+1}\right)^a (|s|+1)^\varepsilon,$$

$$\left\|T(t,s)^{-1}Q(t)\right\| \leq K\left(\frac{|t|+1}{|s|+1}\right)^{-b} (|t|+1)^\varepsilon$$

whenever $0 \geq t \geq s$.

Again the unstable manifolds are obtained as graphs of Lipschitz functions. To describe the class of functions, for each $s \leq 0$ let $B_s^u(\varrho) \subset F(s)$ be the open ball of radius ϱ centered at zero, and take again β as in (10). Given $\delta > 0$, we consider the set of initial conditions

$$Z_\beta^u = Z_\beta^u(\delta) = \left\{(s,\xi) : s \leq 0,\ \xi \in B_s^u\left(\delta(|s|+1)^{-\beta}\right)\right\},$$

and we denote by X_β^u the space of functions $\Phi^u \colon Z_\beta^u \to X$ that are left-continuous in s, at most with discontinuities of the first kind at the points τ_i, such that

$$\Phi^u(s,0) = 0, \qquad \Phi^u\left(s, B_s^u\left(\delta\left(|s|+1\right)^{-\beta}\right)\right) \subset E(s),$$

and satisfying (11) for every $s \leq 0$ and $\xi_1, \xi_2 \in B_s^u(\delta(|s|+1)^{-\beta})$. For each $\Phi^u \in X_\beta^u$ we consider the graph

$$\mathcal{W}^u = \left\{\left(s, \Phi^u(s,\xi), \xi\right) : (s,\xi) \in Z_\beta^u\right\},$$

and we continue to consider the semiflow Ψ_κ defined by (12).

The following is our unstable manifold theorem.

Theorem 3 *Assume that* (1) *admits a nonuniform polynomial dichotomy in* $\mathbb{R}^-$. *If* $b - \beta > 0$, *then there exist constants* $\delta > 0$ *and* $\eta > 1$, *and a unique function* $\Phi^u \in X_\beta^u$ *such that*

$$\Psi_\kappa\left(s, \Phi^u(s,\xi), \xi\right) \in \mathcal{W}^u \quad \textit{for every} \quad (s,\xi) \in Z_{\beta+\varepsilon}^u(\delta/\eta),\ \kappa \geq 0.$$

Furthermore, there exists a constant $d > 0$ *such that*

$$\left\|\Psi_\kappa\left(s, \Phi^u(s,\xi_1), \xi_1\right) - \Psi_\kappa\left(s, \Phi^u(s,\xi_2), \xi_2\right)\right\| \leq d\left(1 + \frac{|\kappa|}{|s|+1}\right)^a (|s|+1)^\varepsilon \|\xi_1 - \xi_2\|$$

for every $\kappa \leq 0$ *and* $(s,\xi_1), (s,\xi_2) \in Z_{\beta+\varepsilon}^u(\delta/\eta)$.

The proof of Theorem 3 is entirely analogous to the proof of Theorem 1 and can be obtained by reversing the time.

5 Polynomial Lyapunov Exponents and Invariant Manifolds

We describe in this section the relation between the notion of nonuniform polynomial dichotomy and an appropriate notion of Lyapunov exponent, in the particular case of finite-dimensional spaces. We also describe the consequences of this relation for the existence of stable invariant manifolds.

5.1 Lyapunov Exponents

We consider in this section the linear impulsive equation (1) in the finite-dimensional space $X=\mathbb{R}^n$. Following [7], we define a function $\chi\colon \mathbb{R}^n \to [-\infty,\infty]$ by

$$\chi(x)=\limsup_{t\to+\infty}\frac{\log\|x(t)\|}{\log t},$$

where $x(t)$ is the solution of (1) with $x(0)=x_0$ (with the convention that $\log 0=-\infty$), and we call it the *polynomial Lyapunov exponent* associated to the linear equation (1). Indeed, χ is a Lyapunov exponent (see for example [3]), in the sense that:

1. $\chi(0)=-\infty$;
2. $\chi(cx)=\chi(x)$ for every $x\in\mathbb{R}^n$ and $c\in\mathbb{R}\setminus\{0\}$;
3. $\chi(x+y)\le\max\{\chi(x),\chi(y)\}$ for every $x,y\in\mathbb{R}^n$.

Therefore, it follows from the abstract theory of Lyapunov exponents that χ restricted to $\mathbb{R}^n\setminus\{0\}$ can take at most n values (that can also be $+\infty$ and $-\infty$).

To introduce the notion of regularity coefficient, we consider the adjoint equation

$$\begin{aligned} y' &= -A(t)^*y, \quad t\ge 0,\ t\ne\tau_i,\\ y(\tau_i^+) &= -(B_i^*)(B_i^*-\mathrm{Id})y(\tau_i), \quad i\in\mathbb{N}, \end{aligned} \tag{30}$$

where A^* denotes the transpose of the matrix A. We note that by assumptions the matrices B_i and thus also the matrices B_i^* are invertible. One can show that if x is a solution of (1) and y is a solution of (30), then

$$\langle x(t),y(t)\rangle=\langle x(0),y(0)\rangle \quad \text{for every } t>0.$$

We also define a function $\tilde\chi\colon\mathbb{R}^n\to[-\infty,\infty]$ by

$$\tilde\chi(y)=\limsup_{t\to+\infty}\frac{\log\|y(t)\|}{\log t},$$

where $y(t)$ is the solution of equation (30) with $y(0)=y_0$, and we call it the *polynomial Lyapunov exponent* associated to the linear equation (30). Again, it follows from the abstract theory of Lyapunov exponents that $\tilde\chi$ can take at most n values in $\mathbb{R}^n\setminus\{0\}$.

Now we assume that when restricted to $\mathbb{R}^n\setminus\{0\}$ both Lyapunov exponents χ and $\tilde\chi$ take only finite values. We define the *regularity coefficient* of χ and $\tilde\chi$ (or of (1) and (30)) by

$$\gamma(\chi,\tilde\chi)=\min\max\{\chi(x_i)+\tilde\chi(y_i):1\le i\le n\},$$

where the minimum is taken over all dual bases $x_1,\dots,x_n$ and $y_1,\dots,y_n$ of $\mathbb{R}^n$, that is, all bases such that $\langle x_i,y_j\rangle=\delta_{ij}$ for each i and j (here δ_{ij} is the Kronecker symbol).

5.2 Criterion for Polynomial Dichotomies

We show in this section that any linear equation in block form, with two blocks respectively with negative and positive (polynomial) Lyapunov exponents, admits a nonuniform polynomial dichotomy. Namely, we assume that there is a decomposition $\mathbb{R}^n = E \oplus F$ (independent of t) with respect to which

$$A(t) = \begin{pmatrix} B(t) & 0 \\ 0 & C(t) \end{pmatrix}. \tag{31}$$

We also consider the regularity coefficients

$$\gamma_E = \gamma\big(\chi|E, \tilde{\chi}|E\big) \quad \text{and} \quad \gamma_F = \gamma\big(\chi|F, \tilde{\chi}|F\big).$$

Theorem 4 *If the matrices $A(t)$ have the block form in (31) for each $t \geq 0$, with*

$$\chi(x) < 0 \quad \textit{for } x \in E \setminus \{0\}, \tag{32}$$

and

$$\chi(x) > 0 \quad \textit{for } x \in F \setminus \{0\}, \tag{33}$$

then for each sufficiently small $\delta > 0$, equation (1) admits a nonuniform polynomial dichotomy with $a < 0 < b$ and $\varepsilon = \max\{\gamma_E, \gamma_F\} + \delta$.

The proof of Theorem 4 is analogous to the proof of Theorem 4 in [7], after observing that if $X(t)X(s)^{-1}$ is the evolution operator of (1), then $Y(t)Y(s)^{-1}$, with $Y(t) = [X(t)^*]^{-1}$, is the evolution operator of (30).

5.3 Existence of Invariant Manifolds

Now we combine Theorem 4 with the stable manifold theorem to show that the existence of negative and positive (polynomial) Lyapunov exponents guarantees the existence of stable and unstable invariant manifolds.

We consider (1) in $\mathbb{R}$ as well as (7), again with the functions f and g_i satisfying the conditions (a) and (b) in Sect. 3 (with $\mathbb{R}_0^+$ replaced by $\mathbb{R}$). We continue to use the notations introduced in the former sections. The following result is a consequence of Theorems 1 and 4.

Theorem 5 *Assume that the matrices $A(t)$ have the block form in (31) for each $t \geq 0$, and that they satisfy (32) and (33). If $a + \beta < 0 < b - \beta$, then there exist constants $\delta > 0$ and $\eta > 1$, and a unique function $\Phi \in X_\beta$ such that (15) and (16) hold.*

We can also formulate a corresponding unstable manifold theorem.

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